Chapter 1

RELATIONS AND FUNCTIONS

1.1 Overview

1.1.1 *Relation*

A relation R from a non-empty set A to a non empty set B is a subset of the Cartesian product A × B. The set of all first elements of the ordered pairs in a relation R from a set A to a set B is called the domain of the relation R. The set of all second elements in a relation R from a set A to a set B is called the range of the relation R. The whole set B is called the codomain of the relation R. Note that range is always a subset of codomain.

1.1.2 *Types of Relations*

A relation R in a set A is subset of A × A. Thus empty set ϕ and A × A are two extreme relations.

(i) A relation R in a set A is called empty relation, if no element of A is related to any element of A, i.e., $R = \phi \subset A \times A$.

(ii) A relation R in a set A is called universal relation, if each element of A is related to every element of A, i.e., R = A × A.

(iii) A relation R in A is said to be reflexive if aRa for all $a \in$ A, R is symmetric if $a\text{R}b \Rightarrow b\text{R}a$, $\forall\ a, b \in$ A and it is said to be transitive if aRb and bR$c \Rightarrow a$Rc $\forall\ a, b, c \in$ A. Any relation which is reflexive, symmetric and transitive is called an equivalence relation.

Note: An important property of an equivalence relation is that it divides the set into pairwise disjoint subsets called equivalent classes whose collection is called a partition of the set. Note that the union of all equivalence classes gives the whole set.

1.1.3 *Types of Functions*

(i) A function $f : X \to Y$ is defined to be one-one (or injective), if the images of distinct elements of X under f are distinct, i.e.,

$$x_1, x_2 \in X, f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$$

(ii) A function $f : X \to Y$ is said to be onto (or surjective), if every element of Y is the image of some element of X under f, i.e., for every $y \in$ Y there exists an element $x \in$ X such that $f(x) = y$.

(iii) A function $f : X \to Y$ is said to be one-one and onto (or bijective), if f is both one-one and onto.

1.1.4 *Composition of Functions*

(i) Let $f : A \to B$ and $g : B \to C$ be two functions. Then, the composition of f and g, denoted by $g \circ f$, is defined as the function $g \circ f : A \to C$ given by

$$g \circ f(x) = g(f(x)), \forall x \in A.$$

(ii) If $f : A \to B$ and $g : B \to C$ are one-one, then $g \circ f : A \to C$ is also one-one

(iii) If $f : A \to B$ and $g : B \to C$ are onto, then $g \circ f : A \to C$ is also onto.

However, converse of above stated results (ii) and (iii) need not be true. Moreover, we have the following results in this direction.

(iv) Let $f : A \to B$ and $g : B \to C$ be the given functions such that $g \circ f$ is one-one. Then f is one-one.

(v) Let $f : A \to B$ and $g : B \to C$ be the given functions such that $g \circ f$ is onto. Then g is onto.

1.1.5 *Invertible Function*

(i) A function $f : X \to Y$ is defined to be invertible, if there exists a function $g : Y \to X$ such that $g \circ f = I_x$ and $f \circ g = I_Y$. The function g is called the inverse of f and is denoted by f^{-1}.

(ii) A function $f : X \to Y$ is invertible if and only if f is a bijective function.

(iii) If $f : X \to Y$, $g : Y \to Z$ and $h : Z \to S$ are functions, then $h \circ (g \circ f) = (h \circ g) \circ f$.

(iv) Let $f : X \to Y$ and $g : Y \to Z$ be two invertible functions. Then $g \circ f$ is also invertible with $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

1.1.6 *Binary Operations*

(i) A binary operation $*$ on a set A is a function $* : A \times A \to A$. We denote $*(a, b)$ by $a * b$.

(ii) A binary operation $*$ on the set X is called commutative, if $a * b = b * a$ for every $a, b \in X$.

(iii) A binary operation $* : A \times A \to A$ is said to be associative if $(a * b) * c = a * (b * c)$, for every $a, b, c \in A$.

(iv) Given a binary operation $* : A \times A \to A$, an element $e \in A$, if it exists, is called identity for the operation $*$, if $a * e = a = e * a$, $\forall a \in A$.

(v) Given a binary operation $*: A \times A \to A$, with the identity element e in A, an element $a \in A$, is said to be invertible with respect to the operation $*$, if there exists an element b in A such that $a * b = e = b * a$ and b is called the inverse of a and is denoted by a^{-1}.

1.2 Solved Examples

Short Answer (S.A.)

Example 1 Let A = {0, 1, 2, 3} and define a relation R on A as follows:

R = {(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (2, 2), (3, 0), (3, 3)}.

Is R reflexive? symmetric? transitive?

Solution R is reflexive and symmetric, but not transitive since for $(1, 0) \in R$ and $(0, 3) \in R$ whereas $(1, 3) \notin R$.

Example 2 For the set A = {1, 2, 3}, define a relation R in the set A as follows:

R = {(1, 1), (2, 2), (3, 3), (1, 3)}.

Write the ordered pairs to be added to R to make it the smallest equivalence relation.

Solution (3, 1) is the single ordered pair which needs to be added to R to make it the smallest equivalence relation.

Example 3 Let R be the equivalence relation in the set **Z** of integers given by R = {(a, b) : 2 divides $a - b$}. Write the equivalence class [0].

Solution $[0] = \{0, \pm 2, \pm 4, \pm 6, ...\}$

Example 4 Let the function $f: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 4x - 1$, $\forall\ x \in \mathbf{R}$. Then, show that f is one-one.

Solution For any two elements $x_1, x_2 \in \mathbf{R}$ such that $f(x_1) = f(x_2)$, we have

$$4x_1 - 1 = 4x_2 - 1$$

$$\Rightarrow \quad 4x_1 = 4x_2, \text{ i.e., } x_1 = x_2$$

Hence f is one-one.

Example 5 If $f = \{(5, 2), (6, 3)\}$, $g = \{(2, 5), (3, 6)\}$, write $f \circ g$.

Solution $f \circ g = \{(2, 2), (3, 3)\}$

Example 6 Let $f: \mathbf{R} \to \mathbf{R}$ be the function defined by $f(x) = 4x - 3$ $\forall\ x \in \mathbf{R}$. Then write f^{-1}.

Solution Given that $f(x) = 4x - 3 = y$ (say), then

$$4x = y + 3$$

$$\Rightarrow \quad x = \frac{y+3}{4}$$

Hence $f^{-1}(y) = \frac{y+3}{4} \Rightarrow f^{-1}(x) = \frac{x+3}{4}$

Example 7 Is the binary operation $*$ defined on **Z** (set of integer) by $m * n = m - n + mn \ \forall\, m, n \in \mathbf{Z}$ commutative?

Solution No. Since for $1, 2 \in \mathbf{Z}$, $1 * 2 = 1 - 2 + 1.2 = 1$ while $2 * 1 = 2 - 1 + 2.1 = 3$ so that $1 * 2 \neq 2 * 1$.

Example 8 If $f = \{(5, 2), (6, 3)\}$ and $g = \{(2, 5), (3, 6)\}$, write the range of f and g.

Solution The range of $f = \{2, 3\}$ and the range of $g = \{5, 6\}$.

Example 9 If A = {1, 2, 3} and f, g are relations corresponding to the subset of A × A indicated against them, which of f, g is a function? Why?

$$f = \{(1, 3), (2, 3), (3, 2)\}$$

$$g = \{(1, 2), (1, 3), (3, 1)\}$$

Solution f is a function since each element of A in the first place in the ordered pairs is related to only one element of A in the second place while g is not a function because 1 is related to more than one element of A, namely, 2 and 3.

Example 10 If A = $\{a, b, c, d\}$ and $f = \{a, b), (b, d), (c, a), (d, c)\}$, show that f is one-one from A onto A. Find f^{-1}.

Solution f is one-one since each element of A is assigned to distinct element of the set A. Also, f is onto since f(A) = A. Moreover, $f^{-1} = \{(b, a), (d, b), (a, c), (c, d)\}$.

Example 11 In the set **N** of natural numbers, define the binary operation $*$ by $m * n = g.c.d\,(m, n)$, $m, n \in \mathbf{N}$. Is the operation $*$ commutative and associative?

Solution The operation is clearly commutative since

$$m * n = g.c.d\,(m, n) = g.c.d\,(n, m) = n * m \quad \forall\, m, n \in \mathbf{N}.$$

It is also associative because for $l, m, n \in \mathbf{N}$, we have

$$l * (m * n) = g.\ c.\ d\,(l, g.c.d\,(m, n))$$

$$= g.c.d.\,(g.\ c.\ d\,(l, m), n)$$

$$= (l * m) * n.$$

Long Answer (L.A.)

Example 12 In the set of natural numbers **N**, define a relation R as follows: $\forall\, n, m \in$ **N**, nRm if on division by 5 each of the integers n and m leaves the remainder less than 5, i.e. one of the numbers 0, 1, 2, 3 and 4. Show that R is equivalence relation. Also, obtain the pairwise disjoint subsets determined by R.

Solution R is reflexive since for each $a \in$ **N**, aRa. R is symmetric since if aRb, then bRa for $a, b \in$ **N**. Also, R is transitive since for $a, b, c \in$ **N**, if aRb and bRc, then aRc. Hence R is an equivalence relation in **N** which will partition the set **N** into the pairwise disjoint subsets. The equivalent classes are as mentioned below:

$A_0 = \{5, 10, 15, 20 ...\}$

$A_1 = \{1, 6, 11, 16, 21 ...\}$

$A_2 = \{2, 7, 12, 17, 22, ...\}$

$A_3 = \{3, 8, 13, 18, 23, ...\}$

$A_4 = \{4, 9, 14, 19, 24, ...\}$

It is evident that the above five sets are pairwise disjoint and

$$A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4 = \bigcup_{i=0}^{4} A_i = \mathbf{N}.$$

Example 13 Show that the function f : **R** → **R** defined by $f(x) = \dfrac{x}{x^2+1}, \forall\, x \in \mathbf{R}$, is neither one-one nor onto.

Solution For $x_1, x_2 \in$ **R**, consider

$$f(x_1) = f(x_2)$$

$$\Rightarrow \frac{x_1}{x_1^2+1} = \frac{x_2}{x_2^2+1}$$

$$\Rightarrow x_1 x_2^2 + x_1 = x_2 x_1^2 + x_2$$

$$\Rightarrow x_1 x_2 (x_2 - x_1) = x_2 - x_1$$

$$\Rightarrow x_1 = x_2 \text{ or } x_1 x_2 = 1$$

We note that there are point, x_1 and x_2 with $x_1 \neq x_2$ and $f(x_1) = f(x_2)$, for instance, if we take $x_1 = 2$ and $x_2 = \frac{1}{2}$, then we have $f(x_1) = \frac{2}{5}$ and $f(x_2) = \frac{2}{5}$ but $2 \neq \frac{1}{2}$. Hence f is not one-one. Also, f is not onto for if so then for $1 \in \mathbf{R}\ \exists\ x \in \mathbf{R}$ such that $f(x) = 1$