Appendix 2

MATHEMATICAL MODELLING

A.2.1 Introduction

In class XI, we have learnt about mathematical modelling as an attempt to study some part (or form) of some real-life problems in mathematical terms, i.e., the conversion of a physical situation into mathematics using some suitable conditions. Roughly speaking mathematical modelling is an activity in which we make models to describe the behaviour of various phenomenal activities of our interest in many ways using words, drawings or sketches, computer programs, mathematical formulae etc.

In earlier classes, we have observed that solutions to many problems, involving applications of various mathematical concepts, involve mathematical modelling in one way or the other. Therefore, it is important to study mathematical modelling as a separate topic.

In this chapter, we shall further study mathematical modelling of some real-life problems using techniques/results from matrix, calculus and linear programming.

A.2.2 Why Mathematical Modelling?

Students are aware of the solution of word problems in arithmetic, algebra, trigonometry and linear programming etc. Sometimes we solve the problems without going into the physical insight of the situational problems. Situational problems need physical insight that is **introduction** of physical laws and some symbols to compare the mathematical results obtained with practical values. To solve many problems faced by us, we need a technique and this is what is known as *mathematical modelling.* Let us consider the following problems:

(i) To find the width of a river (particularly, when it is difficult to cross the river).

(ii) To find the optimal angle in case of shot-put (by considering the variables such as : the height of the thrower, resistance of the media, acceleration due to gravity etc.).

(iii) To find the height of a tower (particularly, when it is not possible to reach the top of the tower).

(iv) To find the temperature at the surface of the Sun.

(v) Why heart patients are not allowed to use lift? (without knowing the physiology of a human being).

(vi) To find the mass of the Earth.

(vii) Estimate the yield of pulses in India from the standing crops (a person is not allowed to cut all of it).

(viii) Find the volume of blood inside the body of a person (a person is not allowed to bleed completely).

(ix) Estimate the population of India in the year 2020 (a person is not allowed to wait till then).

All of these problems can be solved and infact have been solved with the help of Mathematics using mathematical modelling. In fact, you might have studied the methods for solving some of them in the present textbook itself. However, it will be instructive if you first try to solve them yourself and that too without the help of Mathematics, if possible, you will then appreciate the power of Mathematics and the need for mathematical modelling.

A.2.3 Principles of Mathematical Modelling

Mathematical modelling is a principled activity and so it has some principles behind it. These principles are almost philosophical in nature. Some of the basic principles of mathematical modelling are listed below in terms of instructions:

(i) Identify the need for the model. (for what we are looking for)

(ii) List the parameters/variables which are required for the model.

(iii) Identify the available relevent data. (what is given?)

(iv) Identify the circumstances that can be applied (assumptions)

(v) Identify the governing physical principles.

(vi) Identify

- (a) the equations that will be used.
- (b) the calculations that will be made.
- (c) the solution which will follow.

(vii) Identify tests that can check the

- (a) consistency of the model.
- (b) utility of the model.

(viii) Identify the parameter values that can improve the model.

The above principles of mathematical modelling lead to the following: steps for mathematical modelling.

Step 1: Identify the physical situation.

Step 2: Convert the physical situation into a mathematical model by introducing parameters / variables and using various known physical laws and symbols.

Step 3: Find the solution of the mathematical problem.

Step 4: Interpret the result in terms of the original problem and compare the result with observations or experiments.

Step 5: If the result is in good agreement, then accept the model. Otherwise modify the hypotheses / assumptions according to the physical situation and go to Step 2.

The above steps can also be viewed through the following diagram:

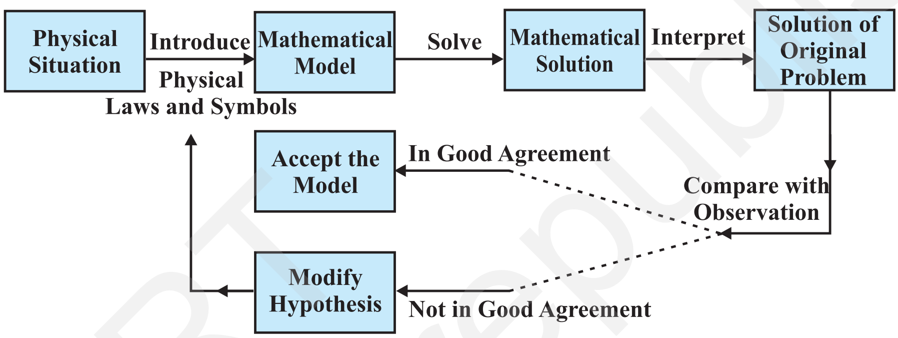

Fig A.2.1

Example 1 Find the height of a given tower using mathematical modelling.

Solution Step 1 Given physical situation is "to find the height of a given tower".

Step 2 Let AB be the given tower (Fig A.2.2). Let PQ be an observer measuring the height of the tower with his eye at P. Let PQ = h and let height of tower be H. Let α be the angle of elevation from the eye of the observer to the top of the tower.

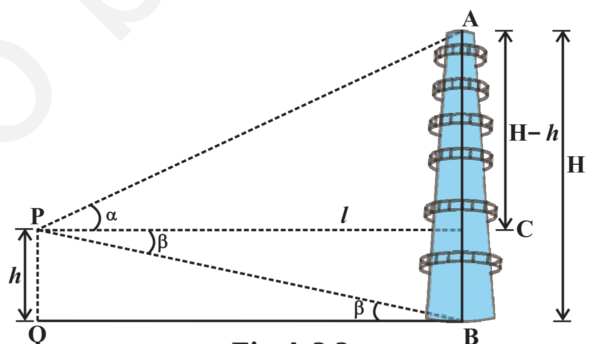

Fig A.2.2

Let $$l = PC = QB$$

Now $$\tan\alpha = \frac{AC}{PC} = \frac{H-h}{l}$$

or $$H = h + l\tan\alpha \qquad \text{... (1)}$$

Step 3 Note that the values of the parameters h, l and α (using sextant) are known to the observer and so (1) gives the solution of the problem.

Step 4 In case, if the foot of the tower is not accessible, i.e., when l is not known to the observer, let β be the angle of depression from P to the foot B of the tower. So from ΔPQB, we have

$$\tan\beta = \frac{PQ}{QB} = \frac{h}{l} \text{ or } l = h\cot\beta$$

Step 5 is not required in this situation as exact values of the parameters h, l, α and β are known.

Example 2 Let a business firm produces three types of products P_1, P_2 and P_3 that uses three types of raw materials R_1, R_2 and R_3. Let the firm has purchase orders from two clients F_1 and F_2. Considering the situation that the firm has a limited quantity of R_1, R_2 and R_3, respectively, prepare a model to determine the quantities of the raw material R_1, R_2 and R_3 required to meet the purchase orders.

Solution Step 1 The physical situation is well identified in the problem.

Step 2 Let A be a matrix that represents purchase orders from the two clients F_1 and F_2. Then, A is of the form

$$A = \begin{array}{c} \\ F_1 \\ F_2 \end{array} \begin{array}{c} \begin{array}{ccc} P_1 & P_2 & P_3 \end{array} \\ \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} \end{array}$$

Let B be the matrix that represents the amount of raw materials R_1, R_2 and R_3, required to manufacture each unit of the products P_1, P_2 and P_3. Then, B is of the form

$$B = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array} \begin{array}{c} \begin{array}{ccc} R_1 & R_2 & R_3 \end{array} \\ \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} \end{array}$$

Step 3 Note that the product (which in this case is well defined) of matrices A and B is given by the following matrix

$$\mathrm{AB} = \begin{array}{c} \\ \mathrm{F}_1 \\ \mathrm{F}_2 \end{array}\begin{array}{c} \begin{array}{ccc} \mathrm{R}_1 & \mathrm{R}_2 & \mathrm{R}_3 \end{array} \\ \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} \end{array}$$

which in fact gives the desired quantities of the raw materials R_1, R_2 and R_3 to fulfill the purchase orders of the two clients F_1 and F_2.

Example 3 Interpret the model in Example 2, in case

$$\mathrm{A} = \begin{bmatrix} 10 & 15 & 6 \\ 10 & 20 & 0 \end{bmatrix}, \ \mathrm{B} = \begin{bmatrix} 3 & 4 & 0 \\ 7 & 9 & 3 \\ 5 & 12 & 7 \end{bmatrix}$$

and the available raw materials are 330 units of R_1, 455 units of R_2 and 140 units of R_3.

Solution Note that

$$\mathrm{AB} = \begin{bmatrix} 10 & 15 & 6 \\ 10 & 20 & 0 \end{bmatrix}\begin{bmatrix} 3 & 4 & 0 \\ 7 & 9 & 3 \\ 5 & 12 & 7 \end{bmatrix}$$

$$= \begin{array}{cccc} & \mathrm{R}_1 & \mathrm{R}_2 & \mathrm{R}_3 \\ \mathrm{F}_1 & 165 & 247 & 87 \\ \mathrm{F}_2 & 170 & 220 & 60 \end{array}$$

This clearly shows that to meet the purchase order of F_1 and F_2, the raw material required is 335 units of R_1, 467 units of R_2 and 147 units of R_3 which is much more than the available raw material. Since the amount of raw material required to manufacture each unit of the three products is fixed, we can either ask for an increase in the available raw material or we may ask the clients to reduce their orders.

Remark If we replace A in Example 3 by A_1 given by

$$\mathrm{A}_1 = \begin{bmatrix} 9 & 12 & 6 \\ 10 & 20 & 0 \end{bmatrix}$$

i.e., if the clients agree to reduce their purchase orders, then

$$\mathrm{A}_1\,\mathrm{B} = \begin{bmatrix} 9 & 12 & 6 \\ 10 & 20 & 0 \end{bmatrix}\begin{bmatrix} 3 & 4 & 0 \\ 7 & 9 & 3 \\ 5 & 12 & 7 \end{bmatrix} \quad \begin{matrix} 141 & 216 & 78 \\ 170 & 220 & 60 \end{matrix}$$