

## INFINITE SERIES

### A.1.1 Introduction

As discussed in the Chapter 9 on Sequences and Series, a sequence  $a_1, a_2, \dots, a_n, \dots$  having infinite number of terms is called *infinite sequence* and its indicated sum, i.e.,  $a_1 + a_2 + a_3 + \dots + a_n + \dots$  is called an *infinite series* associated with infinite sequence. This series can also be expressed in abbreviated form using the sigma notation, i.e.,

$$a_1 + a_2 + a_3 + \dots + a_n + \dots = \sum_{k=1}^{\infty} a_k$$

In this Chapter, we shall study about some special types of series which may be required in different problem situations.

### A.1.2 Binomial Theorem for any Index

In Chapter 8, we discussed the Binomial Theorem in which the index was a positive integer. In this Section, we state a more general form of the theorem in which the index is not necessarily a whole number. It gives us a particular type of infinite series, called *Binomial Series*. We illustrate few applications, by examples.

We know the formula

$$(1+x)^n = {}^n C_0 + {}^n C_1 x + \dots + {}^n C_n x^n$$

Here,  $n$  is non-negative integer. Observe that if we replace index  $n$  by negative integer or a fraction, then the combinations  ${}^n C_r$  do not make any sense.

We now state (without proof), the Binomial Theorem, giving an infinite series in which the index is negative or a fraction and not a whole number.

**Theorem** The formula

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{1.2} x^2 + \frac{m(m-1)(m-2)}{1.2.3} x^3 + \dots$$

holds whenever  $|x| < 1$ .

**Remark** 1. Note carefully the condition  $|x| < 1$ , i.e.,  $-1 < x < 1$  is necessary when  $m$  is negative integer or a fraction. For example, if we take  $x = -2$  and  $m = -2$ , we obtain

$$(1-2)^{-2} = 1 + (-2)(-2) + \frac{(-2)(-3)}{1.2}(-2)^2 + \dots$$

or  $1 = 1 + 4 + 12 + \dots$

This is not possible

2. Note that there are infinite number of terms in the expansion of  $(1+x)^m$ , when  $m$  is a negative integer or a fraction

Consider

$$\begin{aligned} (a+b)^m &= \left[ a \left( 1 + \frac{b}{a} \right) \right]^m = a^m \left( 1 + \frac{b}{a} \right)^m \\ &= a^m \left[ 1 + m \frac{b}{a} + \frac{m(m-1)}{1.2} \left( \frac{b}{a} \right)^2 + \dots \right] \\ &= a^m + ma^{m-1}b + \frac{m(m-1)}{1.2} a^{m-2}b^2 + \dots \end{aligned}$$

This expansion is valid when  $\left| \frac{b}{a} \right| < 1$  or equivalently when  $|b| < |a|$ .

The general term in the expansion of  $(a+b)^m$  is

$$\frac{m(m-1)(m-2)\dots(m-r+1)a^{m-r}b^r}{1.2.3\dots r}$$

We give below certain particular cases of Binomial Theorem, when we assume

$|x| < 1$ , these are left to students as exercises:

1.  $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$
2.  $(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$
3.  $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$
4.  $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$

**Example 1** Expand  $\left(1 - \frac{x}{2}\right)^{-\frac{1}{2}}$ , when  $|x| < 2$ .

**Solution** We have

$$\begin{aligned} \left(1 - \frac{x}{2}\right)^{-\frac{1}{2}} &= 1 + \frac{\left(-\frac{1}{2}\right)}{1} \left(\frac{-x}{2}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1 \cdot 2} \left(\frac{-x}{2}\right)^2 + \dots \\ &= 1 + \frac{x}{4} + \frac{3x^2}{32} + \dots \end{aligned}$$

### A.1.3 Infinite Geometric Series

From Chapter 9, Section 9.5, a sequence  $a_1, a_2, a_3, \dots, a_n$  is called G.P., if  $\frac{a_{k+1}}{a_k} = r$  (constant) for  $k = 1, 2, 3, \dots, n-1$ . Particularly, if we take  $a_1 = a$ , then the resulting sequence  $a, ar, ar^2, \dots, ar^{n-1}$  is taken as the standard form of G.P., where  $a$  is first term and  $r$ , the common ratio of G.P.

Earlier, we have discussed the formula to find the sum of finite series  $a + ar + ar^2 + \dots + ar^{n-1}$  which is given by

$$S_n = \frac{a(1-r^n)}{1-r}.$$

In this section, we state the formula to find the sum of infinite geometric series  $a + ar + ar^2 + \dots + ar^{n-1} + \dots$  and illustrate the same by examples.

Let us consider the G.P.  $1, \frac{2}{3}, \frac{4}{9}, \dots$

Here  $a = 1, r = \frac{2}{3}$ . We have

$$S_n = \frac{1 - \left(\frac{2}{3}\right)^n}{1 - \frac{2}{3}} = 3 \left[ 1 - \left(\frac{2}{3}\right)^n \right] \quad \dots (1)$$

Let us study the behaviour of  $\left(\frac{2}{3}\right)^n$  as  $n$  becomes larger and larger.

$n$	1	5	10	20
$\left(\frac{2}{3}\right)^n$	0.6667	0.1316872428	0.01734152992	0.00030072866

We observe that as  $n$  becomes larger and larger,  $\left(\frac{2}{3}\right)^n$  becomes closer and closer to zero. Mathematically, we say that as  $n$  becomes sufficiently large,  $\left(\frac{2}{3}\right)^n$  becomes sufficiently small. In other words, as  $n \rightarrow \infty$ ,  $\left(\frac{2}{3}\right)^n \rightarrow 0$ . Consequently, we find that the sum of infinitely many terms is given by  $S = 3$ .

Thus, for infinite geometric progression  $a, ar, ar^2, \dots$ , if numerical value of common ratio  $r$  is less than 1, then

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

In this case,  $r^n \rightarrow 0$  as  $n \rightarrow \infty$  since  $|r| < 1$  and then  $\frac{ar^n}{1-r} \rightarrow 0$ . Therefore,

$$S_n \rightarrow \frac{a}{1-r} \text{ as } n \rightarrow \infty.$$

Symbolically, sum to infinity of infinite geometric series is denoted by  $S$ . Thus,

we have  $S = \frac{a}{1-r}$

For example

(i)  $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots = \frac{1}{1 - \frac{1}{2}} = 2$

(ii)  $1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \dots = \frac{1}{1 - \left(-\frac{1}{2}\right)} = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3}$

**Example 2** Find the sum to infinity of the G.P. ;

$$\frac{-5}{4}, \frac{5}{16}, \frac{-5}{64}, \dots$$

**Solution** Here  $a = \frac{-5}{4}$  and  $r = -\frac{1}{4}$ . Also  $|r| < 1$ .

$$\text{Hence, the sum to infinity is } \frac{\frac{-5}{4}}{1 + \frac{1}{4}} = \frac{-5}{\frac{5}{4}} = -4.$$

### A.1.4 Exponential Series

Leonhard Euler (1707 – 1783), the great Swiss mathematician introduced the number  $e$  in his calculus text in 1748. The number  $e$  is useful in calculus as  $\pi$  in the study of the circle.

Consider the following infinite series of numbers

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots \quad \dots (1)$$

The sum of the series given in (1) is denoted by the number  $e$

Let us estimate the value of the number  $e$ .

Since every term of the series (1) is positive, it is clear that its sum is also positive.

Consider the two sums

$$\frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots + \frac{1}{n!} + \dots \quad \dots (2)$$

and  $\frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots + \frac{1}{2^{n-1}} + \dots \quad \dots (3)$

Observe that

$$\frac{1}{3!} = \frac{1}{6} \text{ and } \frac{1}{2^2} = \frac{1}{4}, \text{ which gives } \frac{1}{3!} < \frac{1}{2^2}$$

$$\frac{1}{4!} = \frac{1}{24} \text{ and } \frac{1}{2^3} = \frac{1}{8}, \text{ which gives } \frac{1}{4!} < \frac{1}{2^3}$$

$$\frac{1}{5!} = \frac{1}{120} \text{ and } \frac{1}{2^4} = \frac{1}{16}, \text{ which gives } \frac{1}{5!} < \frac{1}{2^4}.$$