PROOFS IN MATHEMATICS A1

A1.1 Introduction

The ability to reason and think clearly is extremely useful in our daily life. For example, suppose a politician tells you, 'If you are interested in a clean government, then you should vote for me.' What he actually wants you to believe is that if you do not vote for him, then you may not get a clean government. Similarly, if an advertisement tells you, 'The intelligent wear *XYZ* shoes', what the company wants you to conclude is that if you do not wear *XYZ* shoes, then you are not intelligent enough. You can yourself observe that both the above statements may mislead the general public. So, if we understand the process of reasoning correctly, we do not fall into such traps unknowingly.

The correct use of reasoning is at the core of mathematics, especially in constructing proofs. In Class IX, you were introduced to the idea of proofs, and you actually proved many statements, especially in geometry. Recall that a proof is made up of several mathematical statements, each of which is logically deduced from a previous statement in the proof, or from a theorem proved earlier, or an axiom, or the hypotheses. The main tool, we use in constructing a proof, is the process of deductive reasoning.

We start the study of this chapter with a review of what a mathematical statement is. Then, we proceed to sharpen our skills in deductive reasoning using several examples. We shall also deal with the concept of negation and finding the negation of a given statement. Then, we discuss what it means to find the converse of a given statement. Finally, we review the ingredients of a proof learnt in Class IX by analysing the proofs of several theorems. Here, we also discuss the idea of proof by contradiction, which you have come across in Class IX and many other chapters of this book.

A1.2 Mathematical Statements Revisited

Recall, that a 'statement' is a meaningful sentence which is not an order, or an exclamation or a question. For example, 'Which two teams are playing in the

Cricket World Cup Final?' is a question, not a statement. 'Go and finish your homework' is an order, not a statement. 'What a fantastic goal!' is an exclamation, not a statement.

Remember, in general, statements can be one of the following:

- *always true*
- *always false*
- *ambiguous*

In Class IX, you have also studied that in mathematics, **a statement is acceptable only if it is either always true or always false**. So, ambiguous sentences are not considered as mathematical statements.

Let us review our understanding with a few examples.

Example 1 : State whether the following statements are always true, always false or ambiguous. Justify your answers.

(i) The Sun orbits the Earth.
(ii) Vehicles have four wheels.
(iii) The speed of light is approximately 3×10^5 km/s.
(iv) A road to Kolkata will be closed from November to March.
(v) All humans are mortal.

Solution :

(i) This statement is always false, since astronomers have established that the Earth orbits the Sun.

(ii) This statement is ambiguous, because we cannot decide if it is always true or always false. This depends on what the vehicle is — vehicles can have 2, 3, 4, 6, 10, etc., wheels.

(iii) This statement is always true, as verified by physicists.

(iv) This statement is ambiguous, because it is not clear which road is being referred to.

(v) This statement is always true, since every human being has to die some time.

Example 2 : State whether the following statements are true or false, and justify your answers.

(i) All equilateral triangles are isosceles.
(ii) Some isosceles triangles are equilateral.
(iii) All isosceles triangles are equilateral.
(iv) Some rational numbers are integers.

(v) Some rational numbers are not integers.

(vi) Not all integers are rational.

(vii) Between any two rational numbers there is no rational number.

Solution :

(i) This statement is true, because equilateral triangles have equal sides, and therefore are isosceles.

(ii) This statement is true, because those isosceles triangles whose base angles are 60° are equilateral.

(iii) This statement is false. Give a counter-example for it.

(iv) This statement is true, since rational numbers of the form $\frac{p}{q}$, where p is an integer and q = 1, are integers (for example, $3 = \frac{3}{1}$).

(v) This statement is true, because rational numbers of the form $\frac{p}{q}$, p, q are integers and q does not divide p, are not integers (for example, $\frac{3}{2}$).

(vi) This statement is the same as saying 'there is an integer which is not a rational number'. This is false, because all integers are rational numbers.

(vii) This statement is false. As you know, between any two rational numbers r and s lies $\frac{r+s}{2}$, which is a rational number.

Example 3 : If $x < 4$, which of the following statements are true? Justify your answers.

(i) $2x > 8$ (ii) $2x < 6$ (iii) $2x < 8$

Solution :

(i) This statement is false, because, for example, $x = 3 < 4$ does not satisfy $2x > 8$.

(ii) This statement is false, because, for example, $x = 3.5 < 4$ does not satisfy $2x < 6$.

(iii) This statement is true, because it is the same as $x < 4$.

Example 4 : Restate the following statements with appropriate conditions, so that they become true statements:

(i) If the diagonals of a quadrilateral are equal, then it is a rectangle.

(ii) A line joining two points on two sides of a triangle is parallel to the third side.

(iii) $\sqrt{p}$ is irrational for all positive integers p.

(iv) All quadratic equations have two real roots.

Solution :

(i) If the diagonals of a parallelogram are equal, then it is a rectangle.

(ii) A line joining the mid-points of two sides of a triangle is parallel to the third side.

(iii) $\sqrt{p}$ is irrational for all primes p.

(iv) All quadratic equations have at most two real roots.

Remark : There can be other ways of restating the statements above. For instance, (iii) can also be restated as '$\sqrt{p}$ is irrational for all positive integers p which are not a perfect square'.

EXERCISE A1.1

1. State whether the following statements are always true, always false or ambiguous. Justify your answers.

(i) All mathematics textbooks are interesting.

(ii) The distance from the Earth to the Sun is approximately 1.5×10^8 km.

(iii) All human beings grow old.

(iv) The journey from Uttarkashi to Harsil is tiring.

(v) The woman saw an elephant through a pair of binoculars.

2. State whether the following statements are true or false. Justify your answers.

(i) All hexagons are polygons. (ii) Some polygons are pentagons.

(iii) Not all even numbers are divisible by 2. (iv) Some real numbers are irrational.

(v) Not all real numbers are rational.

3. Let a and b be real numbers such that $ab \neq 0$. Then which of the following statements are true? Justify your answers.

(i) Both a and b must be zero. (ii) Both a and b must be non-zero.

(iii) Either a or b must be non-zero.

4. Restate the following statements with appropriate conditions, so that they become true.

(i) If $a^2 > b^2$, then $a > b$. (ii) If $x^2 = y^2$, then $x = y$.

(iii) If $(x + y)^2 = x^2 + y^2$, then $x = 0$. (iv) The diagonals of a quadrilateral bisect each other.

A1.3 Deductive Reasoning

In Class IX, you were introduced to the idea of deductive reasoning. Here, we will work with many more examples which will illustrate how **deductive reasoning** is used to **deduce** conclusions from given statements that we assume to be true. The given statements are called 'premises' or 'hypotheses'. We begin with some examples.

Example 5 : Given that Bijapur is in the state of Karnataka, and suppose Shabana lives in Bijapur. In which state does Shabana live?

Solution : Here we have two premises:

(i) Bijapur is in the state of Karnataka (ii) Shabana lives in Bijapur

From these premises, we deduce that Shabana lives in the state of Karnataka.

Example 6 : Given that all mathematics textbooks are interesting, and suppose you are reading a mathematics textbook. What can we conclude about the textbook you are reading?

Solution : Using the two premises (or hypotheses), we can deduce that you are reading an interesting textbook.

Example 7 : Given that $y = -6x + 5$, and suppose $x = 3$. What is y?

Solution : Given the two hypotheses, we get $y = -6(3) + 5 = -13$.

Example 8 : Given that ABCD is a parallelogram, and suppose AD = 5 cm, AB = 7 cm (see Fig. A1.1). What can you conclude about the lengths of DC and BC?

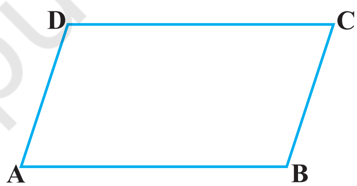

Fig. A1.1

Solution : We are given that ABCD is a parallelogram. So, we deduce that all the properties that hold for a parallelogram hold for ABCD. Therefore, in particular, the property that 'the opposite sides of a parallelogram are equal to each other', holds. Since we know AD = 5 cm, we can deduce that BC = 5 cm. Similarly, we deduce that DC = 7 cm.

Remark : In this example, we have seen how we will often need to find out and use properties hidden in a given premise.

Example 9 : Given that $\sqrt{p}$ is irrational for all primes p, and suppose that 19423 is a prime. What can you conclude about $\sqrt{19423}$?

Solution : We can conclude that $\sqrt{19423}$ is irrational.

In the examples above, you might have noticed that we do not know whether the hypotheses are true or not. We are **assuming** that they are true, and then applying deductive reasoning. For instance, in Example 9, we haven't checked whether 19423