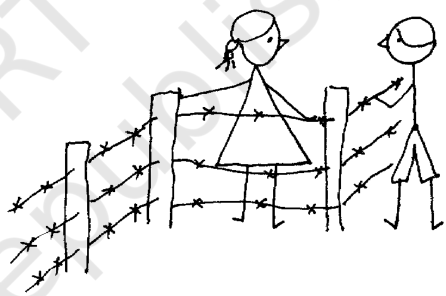
APPENDIX 1

PROOFS IN MATHEMATICS

A1.1 Introduction



Suppose your family owns a plot of land and there is no fencing around it. Your neighbour decides one day to fence off his land. After he has fenced his land, you discover that a part of your family's land has been enclosed by his fence. How will you prove to your neighbour that he has tried to encroach on your land? Your first step may be to seek the help of the village elders to sort out the difference in boundaries. But, suppose opinion is divided among the elders. Some feel you are right and others feel your neighbour is right. What can you do? Your only option is to find a way of establishing your claim for the boundaries of your land that is acceptable to all. For example, a government approved survey map of your village can be used, if necessary in a court of law, to prove (claim) that you are correct and your neighbour is wrong.

Let us look at another situation. Suppose your mother has paid the electricity bill of your house for the month of August, 2005. The bill for September, 2005, however, claims that the bill for August has not been paid. How will you disprove the claim made by the electricity department? You will have to produce a receipt proving that your August bill has been paid.

You have just seen some examples that show that in our daily life we are often called upon to prove that a certain statement or claim is true or false. However, we also accept many statements without bothering to prove them. But, in mathematics we only accept a statement as true or false (except for some axioms) if it has been proved to be so, according to the logic of mathematics.

In fact, proofs in mathematics have been in existence for thousands of years, and they are central to any branch of mathematics. The first known proof is believed to have been given by the Greek philosopher and mathematician Thales. While mathematics was central to many ancient civilisations like Mesopotamia, Egypt, China and India, there is no clear evidence that they used proofs the way we do today.

In this chapter, we will look at what a statement is, what kind of reasoning is involved in mathematics, and what a mathematical proof consists of.

A1.2 Mathematically Acceptable Statements

In this section, we shall try to explain the meaning of a mathematically acceptable statement. A 'statement' is a sentence which is not an order or an exclamatory sentence. And, of course, a statement is not a question! For example,

"What is the colour of your hair?" is not a statement, it is a question.

"Please go and bring me some water." is a request or an order, not a statement.

"What a marvellous sunset!" is an exclamatory remark, not a statement.

However, "The colour of your hair is black" is a statement.

In general, statements can be one of the following:

- *always true*
- *always false*
- *ambiguous*

The word 'ambiguous' needs some explanation. There are two situations which make a statement ambiguous. The first situation is when we cannot decide if the statement is always true or always false. For example, "Tomorrow is Thursday" is ambiguous, since enough of a context is not given to us to decide if the statement is true or false.

The second situation leading to ambiguity is when the statement is subjective, that is, it is true for some people and not true for others. For example, "Dogs are intelligent" is ambiguous because some people believe this is true and others do not.

Example 1 : State whether the following statements are always true, always false or ambiguous. Justify your answers.

(i) There are 8 days in a week.

(ii) It is raining here.

(iii) The sun sets in the west.

(iv) Gauri is a kind girl.

(v) The product of two odd integers is even.

(vi) The product of two even natural numbers is even.

Solution :

(i) This statement is always false, since there are 7 days in a week.

(ii) This statement is ambiguous, since it is not clear where 'here' is.

(iii) This statement is always true. The sun sets in the west no matter where we live.

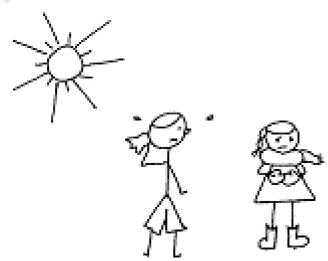
(iv) This statement is ambiguous, since it is subjective–Gauri may be kind to some and not to others.

(v) This statement is always false. The product of two odd integers is always odd.

(vi) This statement is always true. However, to justify that it is true we need to do some work. It will be proved in Section A1.4.

As mentioned before, in our daily life, we are not so careful about the validity of statements. For example, suppose your friend tells you that in July it rains everyday in Manantavadi, Kerala. In all probability, you will believe her, even though it may not have rained for a day or two in July. Unless you are a lawyer, you will not argue with her!

As another example, consider statements we often make to each other like "it is very hot today". We easily accept such statements because we know the context even though these statements are ambiguous. 'It is very hot today' can mean different things to different people because what is very hot for a person from Kumaon may not be hot for a person from Chennai.



But a mathematical statement cannot be ambiguous. *In mathematics, a statement is only acceptable or valid, if it is either true or false.* We say that a statement is true, if it is always true otherwise it is called a false statement.

For example, $5 + 2 = 7$ is always true, so '$5 + 2 = 7$' is a true statement and $5 + 3 = 7$ is a false statement.

Example 2 : State whether the following statements are true or false:

(i) The sum of the interior angles of a triangle is 180°.

(ii) Every odd number greater than 1 is prime.

(iii) For any real number x, $4x + x = 5x$.

(iv) For every real number x, $2x > x$.

(v) For every real number x, $x^2 \geq x$.

(vi) If a quadrilateral has all its sides equal, then it is a square.

Solution :

(i) This statement is true. You have already proved this in Chapter 6.

(ii) This statement is false; for example, 9 is not a prime number.

(iii) This statement is true.

(iv) This statement is false; for example, $2 \times (-1) = -2$, and -2 is not greater than -1.

(v) This statement is false; for example, $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$, and $\frac{1}{4}$ is not greater than $\frac{1}{2}$.

(vi) This statement is false, since a rhombus has equal sides but need not be a square.

You might have noticed that to establish that a statement is not true according to mathematics, all we need to do is to find one case or example where it breaks down. So in (ii), since 9 is not a prime, it is an example that shows that the statement "Every odd number greater than 1 is prime" is not true. Such an example, that counters a statement, is called a *counter-example.* We shall discuss counter-examples in greater detail in Section A1.5.

You might have also noticed that while Statements (iv), (v) and (vi) are false, they can be restated with some conditions in order to make them true.

Example 3 : Restate the following statements with appropriate conditions, so that they become true statements.

(i) For every real number x, $2x > x$.

(ii) For every real number x, $x^2 \geq x$.

(iii) If you divide a number by itself, you will always get 1.

(iv) The angle subtended by a chord of a circle at a point on the circle is 90°.

(v) If a quadrilateral has all its sides equal, then it is a square.

Solution :

(i) If $x > 0$, then $2x > x$.

(ii) If $x \leq 0$ or $x \geq 1$, then $x^2 \geq x$.

(iii) If you divide a number except zero by itself, you will always get 1.

(iv) The angle subtended by a diameter of a circle at a point on the circle is 90°.

(v) If a quadrilateral has all its sides and interior angles equal, then it is a square.

EXERCISE A1.1

1. State whether the following statements are always true, always false or ambiguous. Justify your answers.

(i) There are 13 months in a year.

(ii) Diwali falls on a Friday.

(iii) The temperature in Magadi is 26^0 C.

(iv) The earth has one moon.

(v) Dogs can fly.

(vi) February has only 28 days.

2. State whether the following statements are true or false. Give reasons for your answers.

(i) The sum of the interior angles of a quadrilateral is 350°.

(ii) For any real number x, $x^2 \geq 0$.

(iii) A rhombus is a parallelogram.

(iv) The sum of two even numbers is even.

(v) The sum of two odd numbers is odd.

3. Restate the following statements with appropriate conditions, so that they become true statements.

(i) All prime numbers are odd.

(ii) Two times a real number is always even.

(iii) For any x, $3x + 1 > 4$.

(iv) For any x, $x^3 \geq 0$.

(v) In every triangle, a median is also an angle bisector.

A1.3 Deductive Reasoning

The main logical tool used in establishing the truth of an **unambiguous** statement is *deductive reasoning*. To understand what deductive reasoning is all about, let us begin with a puzzle for you to solve.