Appendix 1

PROOFS IN MATHEMATICS

*❖Proofs are to Mathematics what calligraphy is to poetry.
Mathematical works do consist of proofs just as
poems do consist of characters.
— VLADIMIR ARNOLD ❖*

A.1.1 Introduction

In Classes IX, X and XI, we have learnt about the concepts of a statement, compound statement, negation, converse and contrapositive of a statement; axioms, conjectures, theorems and deductive reasoning.

Here, we will discuss various methods of proving mathematical propositions.

A.1.2 What is a Proof?

Proof of a mathematical statement consists of sequence of statements, each statement being justified with a definition or an axiom or a proposition that is previously established by the method of deduction using only the allowed logical rules.

Thus, each proof is a chain of deductive arguments each of which has its premises and conclusions. Many a times, we prove a proposition directly from what is given in the proposition. But some times it is easier to prove an equivalent proposition rather than proving the proposition itself. This leads to, two ways of proving a proposition directly or indirectly and the proofs obtained are called direct proof and indirect proof and further each has three different ways of proving which is discussed below.

Direct Proof It is the proof of a proposition in which we directly start the proof with what is given in the proposition.

(i) **Straight forward approach** It is a chain of arguments which leads directly from what is given or assumed, with the help of axioms, definitions or already proved theorems, to what is to be proved using rules of logic.

Consider the following example:

Example 1 Show that if $x^2 - 5x + 6 = 0$, then $x = 3$ or $x = 2$.

Solution $x^2 - 5x + 6 = 0$ (given)

$\Rightarrow$ $(x - 3)(x - 2) = 0$ (replacing an expression by an equal/equivalent expression)

$\Rightarrow$ $x - 3 = 0$ or $x - 2 = 0$ (from the established theorem $ab = 0 \Rightarrow$ either $a = 0$ or $b = 0$, for a, b in **R**)

$\Rightarrow$ $x - 3 + 3 = 0 + 3$ or $x - 2 + 2 = 0 + 2$ (adding equal quantities on either side of the equation does not alter the nature of the equation)

$\Rightarrow$ $x + 0 = 3$ or $x + 0 = 2$ (using the identity property of integers under addition)

$\Rightarrow$ $x = 3$ or $x = 2$ (using the identity property of integers under addition)

Hence, $x^2 - 5x + 6 = 0$ implies $x = 3$ or $x = 2$.

Explanation Let p be the given statement "$x^2 - 5x + 6 = 0$" and q be the conclusion statement "$x = 3$ or $x = 2$".

From the statement p, we deduced the statement r: "$(x - 3)(x - 2) = 0$" by replacing the expression $x^2 - 5x + 6$ in the statement p by another expression $(x - 3)(x - 2)$ which is equal to $x^2 - 5x + 6$.

There arise two questions:

(i) How does the expression $(x - 3)(x - 2)$ is equal to the expression $x^2 - 5x + 6$?

(ii) How can we replace an expression with another expression which is equal to the former?

The first one is proved in earlier classes by factorization, i.e.,

$x^2 - 5x + 6 = x^2 - 3x - 2x + 6 = x(x - 3) - 2(x - 3) = (x - 3)(x - 2)$.

The second one is by valid form of argumentation (rules of logic)

Next this statement r becomes premises or given and deduce the statement s "$x - 3 = 0$ or $x - 2 = 0$" and the reasons are given in the brackets.

This process continues till we reach the conclusion.

The symbolic equivalent of the argument is to prove by deduction that $p \Rightarrow q$ is true.

Starting with p, we deduce $p \Rightarrow r \Rightarrow s \Rightarrow \ldots \Rightarrow q$. This implies that "$p \Rightarrow q$" is true.

Example 2 Prove that the function f : **R** $\rightarrow$ **R** defined by $f(x) = 2x + 5$ is one-one.

Solution Note that a function f is one-one if

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \text{ (definition of one-one function)}$$

Now, given that $f(x_1) = f(x_2)$, i.e., $2x_1 + 5 = 2x_2 + 5$

$\Rightarrow$ $2x_1 + 5 - 5 = 2x_2 + 5 - 5$ (adding the same quantity on both sides)

$$\Rightarrow \qquad 2x_1 + 0 = 2x_2 + 0$$

$$\Rightarrow \qquad 2x_1 = 2x_2 \text{ (using additive identity of real number)}$$

$$\Rightarrow \qquad \frac{2}{2}x_1 = \frac{2}{2}x_2 \text{ (dividing by the same non zero quantity)}$$

$$\Rightarrow \qquad x_1 = x_2$$

Hence, the given function is one-one.

(ii) Mathematical Induction

Mathematical induction, is a strategy, of proving a proposition which is deductive in nature. The whole basis of proof of this method depends on the following axiom:

For a given subset S of **N**, if

(i) the natural number $1 \in S$ and

(ii) the natural number $k + 1 \in S$ whenever $k \in S$, then $S = \mathbf{N}$.

According to the principle of mathematical induction, if a statement "S(*n*) is true for *n* = 1" (or for some starting point *j*), and if "S(*n*) is true for *n* = *k*" implies that "S(*n*) is true for *n* = *k* + 1" (whatever integer $k \geq j$ may be), then the statement is true for any positive integer *n*, for all $n \geq j$.

We now consider some examples.

Example 3 Show that if

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \text{ then } A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$$

Solution We have

$$P(n) : A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$$

We note that

$$P(1) : A^1 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Therefore, P(1) is true.

Assume that P(*k*) is true, i.e.,

$$P(k) : A^k = \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$$

We want to prove that P(*k* + 1) is true whenever P(*k*) is true, i.e.,

$$P(k+1) : A^{k+1} = \begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$$

Now $\qquad A^{k+1} = A^k \cdot A$

Since P(*k*) is true, we have

$$A^{k+1} = \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos k\theta\cos\theta - \sin k\theta\sin\theta & \cos k\theta\sin\theta + \sin k\theta\cos\theta \\ -\sin k\theta\cos\theta - \cos k\theta\sin\theta & -\sin k\theta\sin\theta + \cos k\theta\cos\theta \end{bmatrix}$$

(by matrix multiplication)

$$= \begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$$

Thus, P(*k* + 1) is true whenever P(*k*) is true.

Hence, P(*n*) is true for all $n \geq 1$ (by the principle of mathematical induction).

(iii) Proof by cases or by exhaustion

This method of proving a statement $p \Rightarrow q$ is possible only when *p* can be split into several cases, *r*, *s*, *t* (say) so that $p = r \vee s \vee t$ (where "$\vee$" is the symbol for "OR").

If the conditionals $\qquad r \Rightarrow q;$

$\qquad s \Rightarrow q;$

and $\qquad t \Rightarrow q$

are proved, then $(r \vee s \vee t) \Rightarrow q$, is proved and so $p \Rightarrow q$ is proved.

The method consists of examining every possible case of the hypothesis. It is practically convenient only when the number of possible cases are few.

Example 4 Show that in any triangle ABC,

$$a = b\cos C + c\cos B$$

Solution Let *p* be the statement "ABC is any triangle" and *q* be the statement

"$a = b\cos C + c\cos B$"

Let ABC be a triangle. From A draw AD a perpendicular to BC (BC produced if necessary).

As we know that any triangle has to be either acute or obtuse or right angled, we can split *p* into three statements *r*, *s* and *t*, where *r* : ABC is an acute angled triangle with ∠ C is acute.

s : ABC is an obtuse angled triangle with ∠ C is obtuse.

t : ABC is a right angled triangle with ∠ C is right angle.

Hence, we prove the theorem by three cases.

Case (i) When ∠ C is acute (Fig. A1.1).

From the right angled triangle ADB,

$$\frac{BD}{AB} = \cos B$$

i.e. $$BD = AB \cos B = c \cos B$$

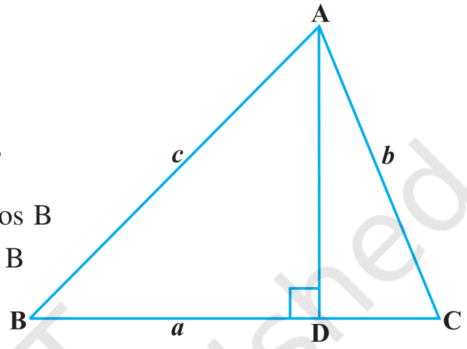

Fig A1.1

From the right angled triangle ADC,

$$\frac{CD}{AC} = \cos C$$

i.e. $$CD = AC \cos C = b \cos C$$

Now $$a = BD + CD = c \cos B + b \cos C \quad \ldots (1)$$

Case (ii) When ∠ C is obtuse (Fig A1.2).

From the right angled triangle ADB,

$$\frac{BD}{AB} = \cos B$$

i.e. $$BD = AB \cos B = c \cos B$$

From the right angled triangle ADC,

$$\frac{CD}{AC} = \cos \angle ACD = \cos (180° - C) = -\cos C$$

i.e. $$CD = -AC \cos C = -b \cos C$$

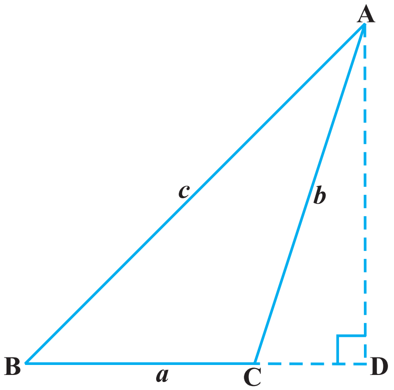

Fig A1.2